Hybrid Higher Order FEM-MoM Analysis of Continuously Inhomogeneous Electromagnetic Scatterers

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***Abstract* — A novel higher order entire-domain finite element technique is presented for accurate and efficient full-wave three-dimensional analysis of electromagnetic structures with continuously inhomogeneous material regions, using large (up to about two wavelengths on a side) generalized curved hierarchical curl-conforming hexahedral vector finite elements (of arbitrary geometrical and field-approximation orders) that allow continuous change of medium parameters throughout their volumes. The results demonstrate considerable reductions in both number of unknowns and computation time of the entire-domain FEM modeling of continuously inhomogeneous materials over piecewise homogeneous models.**

***Keywords* — Computer-aided analysis, electromagnetic analysis, electromagnetic scattering, finite element method, higher order elements, inhomogeneous media, method of moments.**

I. Introduction

IN electromagnetics (EM), the finite element method (FEM) in its various forms and implementations [1]-[4] has been effectively used for quite some time in full-wave three-dimensional (3D) computations based on discretizing partial differential equations. A tremendous amount of effort has been invested in the research of the FEM technique in the past 4 decades, making FEM methodologies and techniques extremely powerful and universal numerical tools for solving a broad range of both closed-region (e.g., waveguide and cavity) and open-region (e.g., antenna and scattering) problems. In the case of open-region problems, hybrid finite element-boundary integral (FE-BI) technique is used for the exact truncation of the unbounded spatial domain [5]-[6].

For modeling and analyzing structures that contain inhomogeneous and complex electromagnetic materials, FEM technique is very efficient and well established as a method of choice. In almost every FEM technique, within a really abundant and impressive body of work in the field, the theory is developed and FEM equations are derived taking advantage of the inherent ability of FEM to directly treat continuously inhomogeneous materials (complex permittivity and permeability of the media can be arbitrary functions, or tensors, of spatial coordinates, e.g., $\varepsilon(\mathbf{r})$ and $\mu(\mathbf{r})$, with $\mathbf{r}$ standing for the position vector of a point in the adopted coordinate system). However, it appears that there are practically no papers on this subject, presenting technique or computer code that actually implements $\varepsilon(\mathbf{r})$ and $\mu(\mathbf{r})$ as continuous space function within a finite element (FE), thus enabling direct computation on finite elements that include arbitrary (continuously) inhomogeneous material. Instead, FEM computations are carried out on piecewise homogeneous approximate model of the inhomogeneous structure, with $\varepsilon(\mathbf{r})$ and $\mu(\mathbf{r})$ replaced by the appropriate piecewise constant approximations. On the other hand, even from the geometrical modeling point of view, it is much simpler and faster to generate a model of the structure with a single, or few, large continuously inhomogeneous elements, than a mesh of a graded layered structure. In most of the practical situations, such an approach can dramatically reduce the time needed for an electromagnetic modeler to set up the problem and initially model the geometry, before any mesher [7] can be used to preprocess the data for analysis.

Numerical modeling employing continuously inhomogeneous finite elements is expected to find practical applications in analysis of a broad range of devices, systems, and phenomena in electromagnetics, including electromagnetic interaction with biological tissues and materials, absorbing coatings for reduction of radar cross sections of targets, scattering and diffraction from inhomogeneous dielectric lenses used for lens antennas and related structures.

For fully exploiting modeling flexibility of continuously inhomogeneous finite elements, these elements should be electrically large, which implies use of the higher order field expansions within the elements, as shown in our preliminary results in [8]. Since the fields in the low order

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FEM technique are approximated by the low-order basis functions, the elements must be electrically very small (on the order of a tenth of the wavelength in each dimension). Subdivision of the structure using such elements results in a discretization of the permittivity and permeability profiles as well, so that elements can be treated as homogeneous, i.e. their treatment as inhomogeneous would practically have no effect on the results. Within the higher order computational approach [9], on the other hand, higher order basis functions enable the use of electrically large geometrical elements (e.g., on the order of a wavelength in each dimension). We refer to the direct FEM computation on such elements as the entire-domain or large-domain analysis. Note that, in general, higher order FEM technique [10]-[14] can greatly reduce the number of unknowns for a given (homogeneous or inhomogeneous) problem and enhance the accuracy and efficiency of the analysis in comparison to the low-order solutions.

II. Theory and Numerical Implementation

Consider an electromagnetic structure that contains some continuously inhomogeneous material regions, as shown in Fig. 1. In our analysis method, the computation domain is first tessellated using higher order geometrical elements in the form of Lagrange-type generalized curved parametric hexahedra of arbitrary geometrical orders K_u, K_v, and K_w ($K_u, K_v, K_w \geq 1$), analytically described as [10]:

$$\mathbf{r}(u,v,w) = \sum_{i=0}^{K_u} \sum_{j=0}^{K_v} \sum_{k=0}^{K_w} \mathbf{r}_{ijk} L_i^{K_u}(u) L_j^{K_v}(v) L_k^{K_w}(w),$$
$$L_i^{K_u}(u) = \prod_{\substack{l=0 \\ l \neq i}}^{K_u} \frac{u - u_l}{u_i - u_l}, \quad -1 \leq u,v,w \leq 1, \tag{1}$$

where $\mathbf{r}_{ijk} = \mathbf{r}(u_i, v_j, w_k)$ are the position vectors of the interpolation nodes, $L_i^{K_u}$ represent Lagrange interpolation polynomials, and similar for $L_j^{K_v}(v)$ and $L_k^{K_w}$. Equation (1) defines a mapping from a cubical parent domain to the generalized hexahedron, as illustrated in Fig. 2.

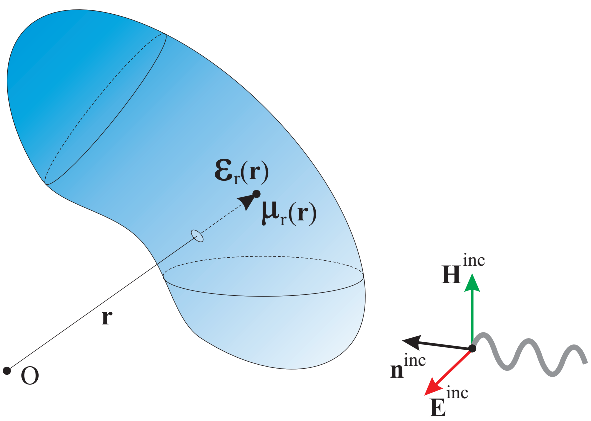

Fig. 1. Electromagnetic structure with continuously inhomogeneous material.

The electric field in the element, $\mathbf{E}(u,v,w)$, is approximated by means of curl-conforming hierarchical polynomial vector basis functions given in [10]; let us denote the functions by $\mathbf{f}(u,v,w)$, and the respective arbitrary field-approximation orders of the polynomial by N_u, N_v, and N_w ($N_u, N_v, N_w \geq 1$). The higher order hierarchical basis functions with improved orthogonality and conditioning properties constructed from Legendre polynomials [12] may also be implemented.

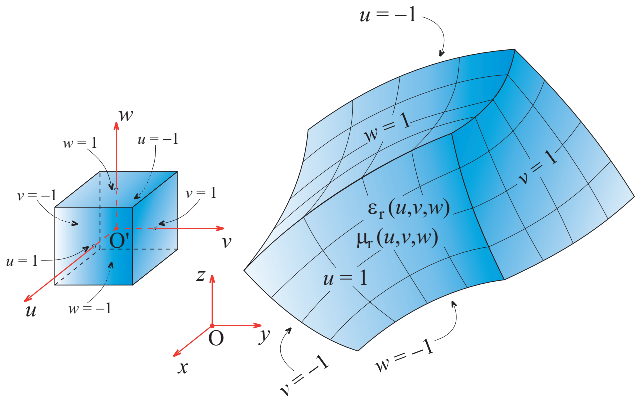

Fig. 2. Generalized curved parametric hexahedron defined by (1), with the continuous variation of medium parameters given by (2); cubical parent domain is also shown.

Continuous variation of medium parameters in the computation model can be implemented in different ways. We choose to take full advantage of the already developed Lagrange interpolating scheme for defining element spatial coordinates in (1), which can be conveniently reused to govern the change of both the complex relative permittivity and permeability, ε_r and μ_r, within the element shown in Fig. 2, as follows:

$$\varepsilon_r(u,v,w) = \sum_{i=0}^{K_u} \sum_{j=0}^{K_v} \sum_{k=0}^{K_w} \varepsilon_{r,ijk} L_i^{K_u}(u) L_j^{K_v}(v) L_k^{K_w}(w), \quad -1 \leq u,v,w \leq 1, \tag{2}$$

where $\varepsilon_{r,ijk} = \varepsilon_r(u_i, v_j, w_k)$ are the relative permittivity values at the point defined by position vectors of spatial interpolation nodes, $\mathbf{r}_{ijk}$, and similarly for μ_r. With such representation of material properties, we then solve for the unknown field coefficients by substituting the field expansion $\mathbf{E}(u,v,w)$ in the curl-curl electric-field vector wave equation [10], reading:

$$\nabla \times \mu_r^{-1}(u,v,w) \nabla \times \mathbf{E}(u,v,w) - k_0^2 \varepsilon_r(u,v,w) \mathbf{E}(u,v,w) = 0, \tag{3}$$

where $k_0 = \omega\sqrt{\varepsilon_0 \mu_0}$ stands for the free-space wave number. A standard Galerkin weak-form discretization of (3) yields:

$$\int_V \mu_r^{-1}(u,v,w) [\nabla \times \mathbf{f}_t(u,v,w)] \cdot [\nabla \times \mathbf{E}(u,v,w)] \, dV - k_0^2 \int_V \varepsilon_r(u,v,w) \mathbf{f}_t(u,v,w) \cdot \mathbf{E}(u,v,w) \, dV = -\oint_S \mu_r^{-1}(u,v,w) \mathbf{f}_t(u,v,w) \cdot \mathbf{n} \times [\nabla \times \mathbf{E}(u,v,w)] \, dS, \tag{4}$$

where V is the volume of the element, bounded by the surface S, $\mathbf{n}$ is the outward unit normal on S, and $\mathbf{f}_t$ are the testing functions (the same as the basis functions). Once the field coefficients are found, all quantities of interest for the analysis are obtained in a straightforward manner.

III. Results and Discussion

As an example of entire-domain FEM analysis of an open-region continuously inhomogeneous structure, consider a lossless cubical dielectric ($\mu_r = 1$) scatterer, of the side length $2a$, and a linear variation of ε_r from $\varepsilon_r = 1$ at the surface to $\varepsilon_r = 6$ at the center of the cube, as shown in Fig. 3. The scatterer is situated in free space and illuminated by a uniform plane wave incident normal to one face of the scatterer, as shown. The FEM domain is truncated at the cube faces by means of unknown electric and magnetic surface currents of densities $\mathbf{J}_S$ and $\mathbf{J}_{mS}$, respectively, that are evaluated by the MoM/SIE, giving rise to a hybrid higher order FEM-MoM solution [6].

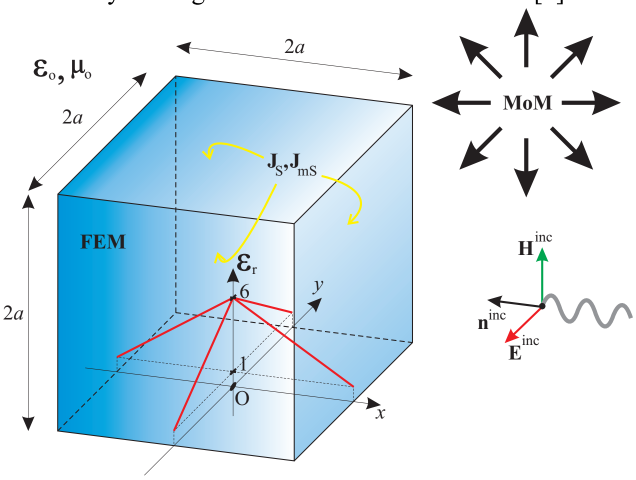

Fig. 3. FEM-MoM analysis of a lossless continuously inhomogeneous cubical dielectric scatterer. Single-element FEM domain with linear variation of permittivity.

To represent this permittivity variation using the expansions in (2), the cube is modeled by 7 trilinear hexahedral finite elements of the first geometrical order. Namely, one small cube-like hexahedral, $\frac{a}{10}$ in length, is situated at the cube center and surrounded by 6 "cushion"-like hexahedra between the central cube and the scatterer surface, onto witch 6 bilinear quadrilateral MoM patches are attached. The obtained model of the scatterer is shown in Fig. 4. The field/current approximation orders are 5 in all directions for all FE "cushions" and 4 in all directions for the central finite element and all MoM patches, resulting in 2560 FEM and 354 MoM unknowns, and a total of 339 s of simulation time for 35 frequencies.

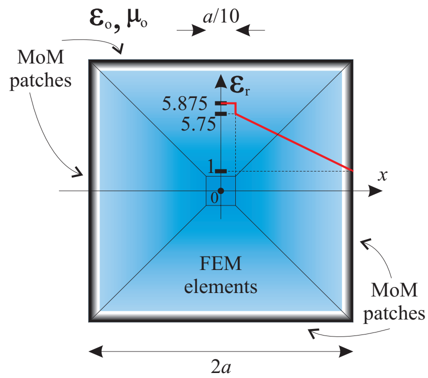

Fig. 4. Model of a lossless, continuously inhomogeneous cubical dielectric scatterer consisting of 7 finite elements and 6 quadrilateral MoM patches.

To both validate the continuously inhomogeneous FEM-MoM model of the scatterer and evaluate its efficiency against the piecewise homogeneous approximate model, the scattering results of the higher order FEM-MoM analysis using large finite elements with continuously changing ε_r are compared with the solution obtained by higher order FEM-MoM simulations of piecewise homogeneous approximate model of the structure in Fig. 4. Each of the 6 "cushions" of the continuous model are replaced, respectively by, $N_l = 2, 3, 4,$ and 7 homogeneous thin "cushions" (plate-like layers), approximating the continuously inhomogeneous profile, which is illustrate in Fig. 5 for $N_l = 4$. Each plate-like layer in layered models is represented by 6 finite elements of the first geometrical order. Field-approximation orders in these elements are 2 in the radial direction and 5 in transversal directions.

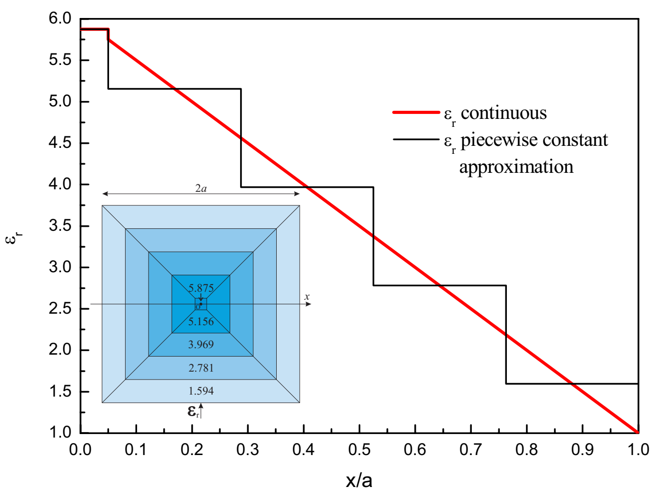

Fig. 5. Piecewise homogeneous approximate graded model ($N_l = 4$) of the structure in Fig. 3, and piecewise constant approximation of relative permittivity profiles.

Shown in Fig. 6 is the normalized (to λ_0^2) monostatic radar cross section (RCS) of the cube (as a function of a/λ_0), λ_0 being the free space wavelength. We observe a very good convergence of the results obtained by the layered FEM-MoM technique to those for the continuous FEM-MoM model, as N_1 increases, as well as an excellent agreement between the 7-layer and continuous FEM-MoM solution. Theoretically, only an infinite number of layers would give the exact solution to the problem in Fig. 3.

As an additional verification of the analysis, the higher order MoM technique based on the surface integral equation (SIE), i.e. MoM/SIE technique [14], is used to obtain a reference solution for the 7-layer model. An excellent agreement of the higher order MoM/SIE and the corresponding FEM-MoM results, for the 7-layer model, is observed.

The additional data shown in the legend of Fig. 6 gives a comparison of the number of unknowns used and the computation time for all 6 models. We conclude from the data shown that the continuous material model is substantially more efficient than the layered analysis. It is 34.16 times faster than the most accurate layered solution (for $N_1 = 7$), and the total number of unknowns is reduced 5.6 times.

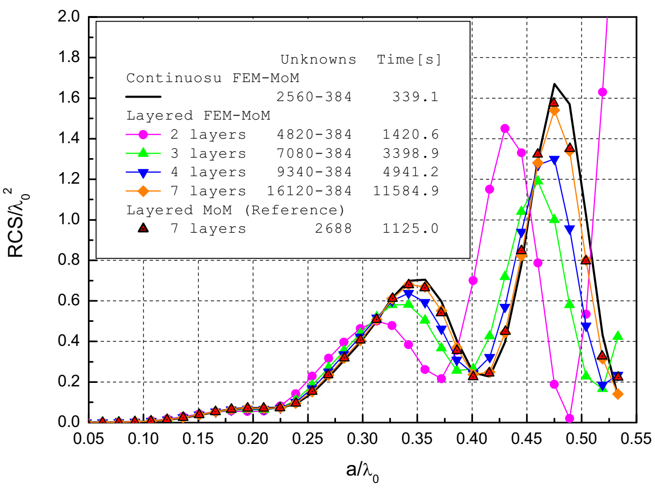

Fig. 6. Normalized monostatic radar cross section of the cubical scatterer in Fig. 3 (λ_0 is the free-space wavelength). Results using the continuous FEM-MoM, four different layered FEM-MoM models, and reference MoM/SIE model are compared.

All numerical results are obtained using HP EliteBook 8440p notebook computer with Intel i5-540 CPU running at 2.53 GHz and with 2 GB of RAM under Microsoft Windows 7 operating system.

IV. Conclusion

A novel higher order entire domain hybrid FEM-MoM technique for accurate and efficient full-wave 3D EM analysis using large (up to two wavelengths on a side) finite elements that allow continuous change of medium parameters throughout their volumes, has been presented. Lagrange-type generalized curved parametric hexahedra of arbitrary geometrical orders with the curl-conforming hierarchical polynomial vector basis functions of arbitrary field-approximation orders and Lagrange interpolation scheme for variations of medium parameters have been used.

The validity, accuracy, and efficiency of the new technique have been demonstrated through an example of EM scatterer with linearly varying permittivity of the dielectric. The example has shown that effective higher order FEM hexahedral meshes, constructed from a very small number of large finite elements with *p*-refined field distributions of high approximation orders, is the method of choice for EM modeling of structures including material inhomogeneities. High efficiency, and considerable reductions in both number of unknowns and computation time of the entire-domain FEM modeling of continuously inhomogeneous material, in comparison with the piecewise homogeneous (layered) models, have been demonstrated.

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