Capacity Bounds and Achievable Rates of the Gaussian Relay Channel

Marjan Rizinski and Venceslav Kafedziski

***Abstract*—We present capacity bounds and achievable rates of the Gaussian relay channel. To evaluate achievable rates we consider two cooperative strategies: decode and forward (DF) and compress and forward (CF). We investigate the impact of different channel conditions of the transmitter-relay, relay-receiver, and transmitter-receiver transmission on the bounds and rates. We also compare the results to the rate of direct transmission to illustrate the benefits of using cooperative strategies in wireless networks.**

***Index Terms*—Relay channel, channel capacity, cooperative strategies, decode and forward, compress and forward, wireless networks.**

I. INTRODUCTION

In traditional wireless communication systems, the physical level is regarded as a point-to-point channel that reliably transmits information from one device to another device. This communication is limited by physical phenomena such as path loss, shadowing and multipath fading which degrade the signal quality of the wireless users. The use of multiple transmit and receive antennas increases the channel capacity and provides diversity to combat these negative effects, but this solution is often impractical, and even impossible, because of the limited size of the wireless devices.

Network communications become more reliable and efficient when network devices help each other to transfer information. To improve the performance of the next generation wireless communication systems and to overcome the size limitation of wireless devices, cooperative transmission is proposed among users that are located close to each other.

Information theory is a powerful tool to analyze cooperative communication scenarios. The information-theoretic background was set by van der Meulen, who introduced the Gaussian relay channel in 1971 [1]. In 1979, Cover and El Gamal derived capacity theorems for the relay channel and presented capacity bounds [2].

The field of cooperative communications has attracted considerable attention since it was shown that cooperation makes systems more robust to the negative effects of fading [3], [4]. The spatial diversity in which diversity gains are achieved via cooperation of wireless users is called cooperative diversity. Different cooperative diversity strategies are known in the literature [5], [6].

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Research has also shown that cooperation increases channel capacity. Cooperation is beneficial and has a capacity advantage over direct transmission even in the "cheap" relay channel, in which the devices operate in half-duplex mode [7]. Upper and lower bounds on the capacity of the Gaussian relay channel are studied in [8], [9]. Some authors analyze the impact of channel state information (CSI) and power allocation on the relay channel capacity [10], [11].

In this paper we focus on the Gaussian relay channel for the fixed channel gain case. We present results on capacity bounds and achievable rates of the Gaussian relay channel for two cooperative strategies, decode and forward, and compress and forward, for both full-duplex and half-duplex mode of transmission. We compare the results obtained under different channel conditions between the transmitter and relay, relay and receiver, and transmitter and receiver. We also investigate the dependence of capacity bounds and achievable rates on the relay position, assuming realistic propagation conditions.

The rest of the paper is organized as follows. Section II describes the channel model and two cooperative strategies: decode and forward and compress and forward. Section III presents capacity bounds of the Gaussian relay channel and achievable rates for the two cooperative strategies. Section IV contains simulation results. Section V concludes the paper.

II. CHANNEL MODEL

The channel model is represented in Fig. 1. The transmitter

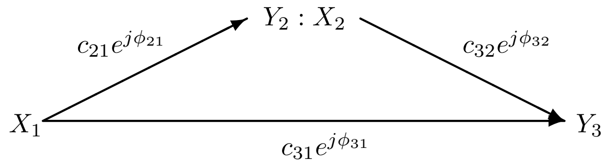

Fig. 1. The Gaussian relay channel.

(node 1) sends a message to the receiver (node 3). The transmission is assisted by a relay (node 2). The powers of the transmitter and the relay are P_1 and P_2 respectively.

The received signals at the relay and the receiver are given by:

$$y_2[i] = c_{21}e^{j\phi_{21}}x_1[i] + z_2[i] \quad (1)$$

$$y_3[i] = c_{31}e^{j\phi_{31}}x_1[i] + c_{32}e^{j\phi_{32}}x_2[i] + z_3[i] \quad (2)$$

where z_2 and z_3 represent AWGN noise with unit variance and $\{c_{ik}e^{j\phi_{ik}}\}$ are the complex channel gains.

We consider full-duplex and half-duplex modes of operation. We say that the relay channel operates in full-duplex mode if the relay node can receive and transmit simultaneously. We say that the relay channel operates in half-duplex mode, if transmission and reception take place in different frequency bands or different time intervals. It is known that the half-duplex mode is the one that can be easily implemented in practical systems [8].

We also consider synchronous and asynchronous channel models. In the synchronous channel model, each destination node (relay and receiver) has complete CSI and knows the magnitude and the phase of the corresponding complex channel gain. In the asynchronous channel model, each destination node knows the magnitude, but not necessarily the phase of the corresponding complex channel gain.

A specific cooperative strategy depends on the way information is processed at the relay node. The optimal information processing at the relay is unknown. Therefore, the determination of the Gaussian relay channel capacity still remains an open problem. In the literature, there are several cooperative strategies that give achievable rates, which present lower bounds on the relay channel capacity. Here, we consider two known cooperative strategies: decode and forward and compress and forward.

In decode and forward (DF), the relay first fully decodes the signal received from the transmitter, re-encodes it and then forwards it to the receiver. The relay might use a different codebook than the transmitter. The receiver decodes the message combining the signals that it receives from the transmitter and the relay [12].

In compress and forward (CF), the relay compresses the signal x_1 received from the transmitter within certain distortion. The received signals at the relay y_2 and at the receiver $y_3^{'}$ are correlated since they are copies of the same signal x_1 obtained from two independent paths with noise and path loss. The relay uses Wyner-Ziv coding to compress y_2 treating $y_3^{'}$ as side information [13]. The compressed signal $\hat{y}_2$ is encoded as $\hat{x}_2$ which is sent to the receiver. $\hat{x}_2$ is received at the receiver as signal $y_3^{''}$. The receiver then combines $y_3^{'}$ and $y_3^{''}$ to decode the source message.

III. Expressions for capacity bounds and achievable rates

In this section, C^+ denotes an upper bound on the channel capacity, and R denotes an achievable rate of a specific cooperative strategy, which also represents a lower bound on the channel capacity.

A. Full-duplex case

Applying the cut-set theorem [14] on the Gaussian relay channel, one may find the upper bound C^+ on the channel capacity, and the achievable rates R_{DF} and R_{CF} of the DF and CF strategies [8]:

$$C^+ = \max_{0\le\beta\le1} \min\Big\{\frac{1}{2}\log(1+(1-\beta)P_1(c_{21}^2+c_{31}^2)),\\ \frac{1}{2}\log\left(1+c_{31}^2P_1+c_{32}^2P_2+2\sqrt{\beta c_{31}^2c_{32}^2P_1P_2}\right)\Big\}. \quad (3)$$

$$R_{DF} = \max_{0\le\beta\le1} \min\Big\{\frac{1}{2}\log(1+(1-\beta)c_{21}^2P_1),\\ \frac{1}{2}\log\left(1+c_{31}^2P_1+c_{32}^2P_2+2\sqrt{\beta c_{31}^2c_{32}^2P_1P_2}\right)\Big\}. \quad (4)$$

$$R_{CF} = \frac{1}{2}\left(1+c_{31}^2P_1+\frac{c_{21}^2P_1}{1+\frac{c_{31}^2P_1+c_{21}^2P_1+1}{c_{32}^2P_2}}\right) \quad (5)$$

The parameter β is a correlation factor between the channel input X_1 and the relay signal X_2. The upper and lower bounds in the asynchronous case can be found by setting $\beta = 0$.

B. Half-duplex case

We consider a half-duplex mode, in which for a given time window D, the relay receives information for a fraction of time αD (relay-receive period) and transmits information in the remaining $(1-\alpha)D$ (relay-transmit period). α is a number that satisfies $0 \le \alpha \le 1$. This mode is known as time-division mode (TD).

We suppose that the transmitter has power $P_1^{(1)}$ and $P_1^{(2)}$ in the relay-receive and relay-transmit period respectively. Then we have the following upper bound C^+ on the channel capacity:

$$C^+ = \max_{0\le\beta\le1} \min\{C_1^+(\beta), C_2^+(\beta)\} \quad (6)$$

where $C_1^+(\beta)$ and $C_2^+(\beta)$ are given by:

$$\begin{aligned} C_1^+(\beta) = {} & \frac{\alpha}{2}\log\left(1+(c_{31}^2+c_{21}^2)P_1^{(1)}\right) \\ & +\frac{1-\alpha}{2}\log\left(1+(1-\beta)c_{31}^2P_1^{(2)}\right) \quad (7) \\ C_2^+(\beta) = {} & \frac{\alpha}{2}\log\left(1+c_{31}^2P_1^{(1)}\right) \\ & +\frac{1-\alpha}{2}\log\Big(1+c_{31}^2P_1^{(2)}+c_{32}^2P_2 \\ & +2\sqrt{\beta c_{31}^2P_1^{(2)}c_{32}^2P_2}\Big) \quad (8) \end{aligned}$$

The achievable rate R_{DF} of the DF strategy is given by:

$$R_{DF} = \max_{0\le\beta\le1} \min\{R_1(\beta), R_2(\beta)\} \quad (9)$$

where $R_1(\beta)$ and $R_2(\beta)$ are given by:

$$\begin{aligned} R_1(\beta) = {} & \frac{\alpha}{2}\log\left(1+c_{21}^2P_1^{(1)}\right) \\ & +\frac{1-\alpha}{2}\log\left(1+(1-\beta)c_{31}^2P_1^{(2)}\right) \quad (10) \\ R_2(\beta) = {} & \frac{\alpha}{2}\log\left(1+c_{31}^2P_1^{(1)}\right) \\ & +\frac{1-\alpha}{2}\log\Big(1+c_{31}^2P_1^{(2)}+c_{32}^2P_2 \\ & +2\sqrt{\beta c_{31}^2P_1^{(2)}c_{32}^2P_2}\Big) \quad (11) \end{aligned}$$

The achievable rate R_{CF} of CF strategy is given by:

$$R_{CF} = \frac{\alpha}{2}\log\left(1 + c_{31}^2 P_1^{(1)} + \frac{c_{21}^2 P_1^{(1)}}{1+\sigma_w^2}\right) + \frac{1-\alpha}{2}\log\left(1 + c_{31}^2 P_1^{(2)}\right) \tag{12}$$

where σ_w^2 is the so-called "compression noise" given by:

$$\sigma_w^2 = \frac{c_{21}^2 P_1^{(1)} + c_{31}^2 P_1^{(1)} + 1}{\left(\left(1 + \frac{c_{32}^2 P_2}{1+c_{31}^2 P_1^{(2)}}\right)^{(1-\alpha)/\alpha} - 1\right)(c_{31}^2 P_1^{(1)} + 1)} \tag{13}$$

The corresponding bounds and rates in the asynchronous case can be found by setting $\beta = 0$ in the previous equations. The achievable rates R_{DF} and R_{CF} also represent lower bounds on the channel capacity.

Assuming that the parameters α, $P_1^{(1)}$, $P_1^{(2)}$ and P_2 are fixed, we can denote the corresponding capacity by $C_R(\alpha, P_1^{(1)}, P_1^{(2)}, P_2)$. Let us suppose that the transmitter and the relay are limited by average power constraints P_1 and P_2. Since the relay transmits information only in the relay-transmit period of length $(1-\alpha)D$, it can use power $\frac{P_2}{1-\alpha}$ during the transmission. Similarly, the transmitter uses power $\frac{\kappa P_1}{\alpha}$ during the relay-receive period and power $\frac{(1-\kappa)P_1}{1-\alpha}$ during the relay-transmit period, where $0 \le \kappa \le 1$ so that the average power constraint is satisfied. With these assumptions, the capacity in the half-duplex case is given by:

$$C_{TD}(P_1, P_2) = \\ = \max_{0\le\alpha\le1, 0\le\kappa\le1} C_R\left(\alpha, \frac{\kappa P_1}{\alpha}, \frac{(1-\kappa)P_1}{1-\alpha}, \frac{P_2}{1-\alpha}\right) \tag{14}$$

The capacity upper bound and the achievable rates are determined when C_R is replaced by the corresponding expressions. The optimization of α and κ is performed numerically.

IV. Simulation results

We investigate the behavior of capacity bounds and achievable rates of the Gaussian relay channel given in the previous section, under various channel scenarios with different channel gains, power constraints, SNR levels and transmitter-relay distances.

Fig. 2 presents capacity bounds and achievable rates in terms of c_{21}^2. DF performs better when the relay is close to the transmitter ($c_{21} \gg c_{32}$), while CF performs better when the relay is close to the receiver ($c_{21} \ll c_{32}$). As c_{21} becomes larger compared to c_{31}, DF rate begins to grow and exceeds CF rate. Increasing the transmitter-relay distance (decreasing c_{21}) reduces the DF achievable rate. When $c_{21} < c_{31}$, both full-duplex and half-duplex DF rates are identical with the rate of direct transmission. CF always gives a rate higher than the rate of direct transmission. Therefore, it can be used for any c_{21}.

Similar conclusions can be drawn from Fig. 3 and Fig. 4, where the capacity bounds and achievable rates are given in terms of SNR. Since the AWGN noise has unit variance, the signal-to-noise ratio is given by $SNR = P_1 c_{31}^2$.

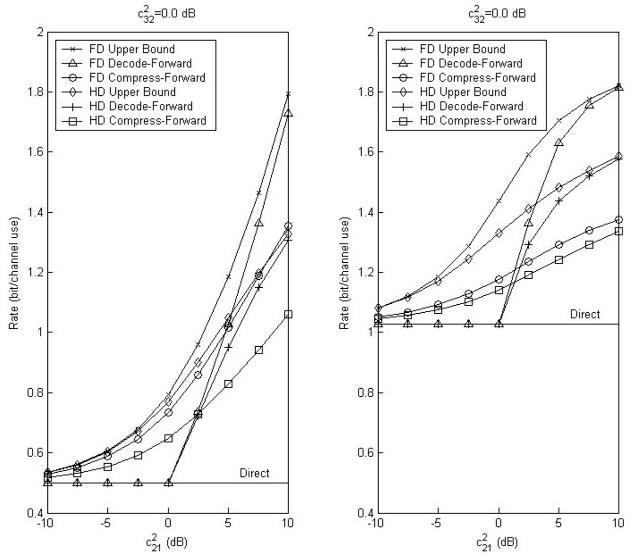

Fig. 2. Capacity bounds and achievable rates in terms of c_{21}^2 for $c_{31}^2 = 0dB$. Left graph: $P_1 = 0dB$, $P_2 = 10dB$. Right graph: $P_1 = 5dB$, $P_2 = 5dB$. "Direct" is the rate of direct transmission. FD and HD stand for full-duplex and half-duplex respectively.

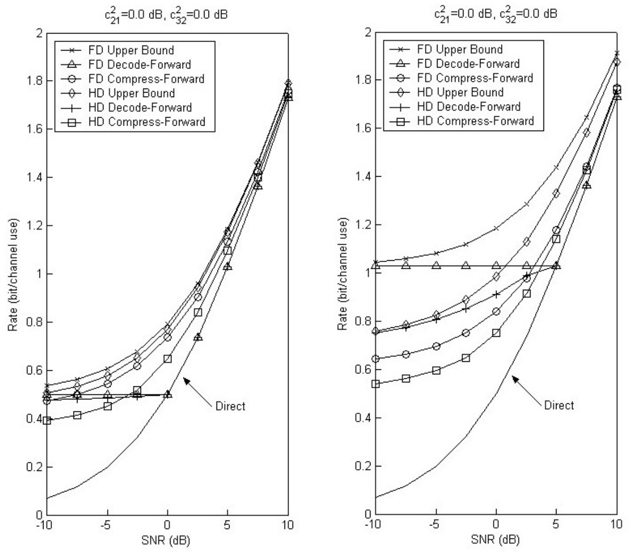

Fig. 3. Capacity bounds and achievable rates in terms of the SNR between the transmitter and the receiver. Left graph: $P_1 = 0dB$, $P_2 = 10dB$. Right graph: $P_1 = 5dB$, $P_2 = 5dB$.

Fig. 3 shows that at low and medium SNR, DF and CF perform better than direct transmission. All the rates for the case $P_1 = P_2 = 5dB$ are higher than the corresponding rates for the case $P_1 = 0dB$, $P_2 = 10dB$. The $P_1 = 0dB$, $P_2 = 10dB$ case results in reduced preformance loss of CF with respect to DF, compatred to the $P_1 = P_2 = 5dB$ case, due to higher value of P_2. At high SNR, the performance of DF and CF strategies tends to approach the performance of direct transmission.

Fig. 4 shows that the rates are close to their corresponding capacity bounds because, in this case, the transmitter has more

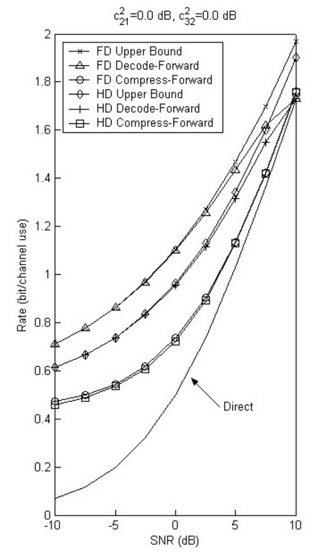

Fig. 4. Capacity bounds and achievable rates in terms of the SNR between the transmitter and the receiver for $P1 = 10dB$ and $P_2 = 0dB$.

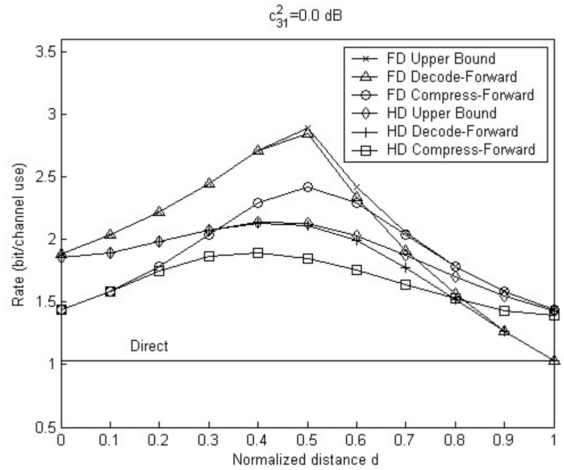

Fig. 5. Capacity bounds and achievable rates in terms of the transmitter-relay distance d for $P_1 = P_2 = 5dB$.

power than the relay. For $SNR < 2dB$, the DF full-duplex and half-duplex rates are close to the FD and HD upper bound on the channel capacity respectively. The DF half-duplex and the DF full-duplex rate depart from their corresponding capacity bounds at high SNR. The CF half-duplex rate closely follows the CF full-duplex rate for all SNR's considered in Fig. 4.

Fig. 5 shows the dependence of the capacity bounds and achievable rates on the transmitter-relay distance d. We consider a scenario where the three nodes lie on a same line and that the transmitter-receiver distance equals 1. We assume $c_{31}^2 = 1$, $c_{21}^2 = \frac{1}{d^\gamma}$ and $c_{32}^2 = \frac{1}{(1-d)^\gamma}$, where γ is the pathloss exponent which depends on terrain and other environmental factors. We adopt the value $\gamma = 4$ which is usually used to model propagation in metropolitan areas. The DF full-duplex and half-duplex rates are very close to the corresponding full-duplex and half-duplex upper capacity bounds for $0 \leq d \leq 0.5$. The CF full-duplex rate is very close to the full-duplex capacity bound for $0.7 \leq d \leq 1$. The DF half-duplex rate approaches the DF full-duplex rate as d approaches 1, where both rates approach the rate of direct transmission. As expected, CF strategy has better performance than DF strategy as d approaches 1. The CF full-duplex rate crosses the DF full-duplex rate at $d = 0.62$. The CF half-duplex rate crosses the DF half-duplex rate at $d = 0.8$.

V. Conclusion

We present capacity bounds and achievable rates of the Gaussian relay channel for two cooperative strategies: decode and forward and compress and forward. We investigate the rate performance under various channel scenarios. Both strategies always results in higher rate compared to the rate of direct transmission. The DF strategy is preferable when the relay is close to the transmitter while the CF strategy is preferable when the relay is close to the receiver. Full-duplex transmission mode gives higher rates than half-duplex transmission mode. The obtained results confirm that cooperative transmission has potential to improve the performance of future generation wireless communication systems.

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