A Robust Estimation of Polynomial-Phase Coefficients

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***Abstract* — We propose a method for the phase estimation of a polynomial-phase signal (PPS). Using an approach based on the high-order instantaneous moment (HIM) of the PPS, the estimation of phase coefficients boils down to the sinusoid frequency estimation. Since the desired complex sinusoid is embedded in a heavy-tailed noise, standard periodogram-based techniques cannot be used for the frequency estimation. Instead of the standard periodogram, we use the robust *M*-periodogram, where a non-quadratic loss function is used for fitting of observations corrupted by noise with unknown heavy-tailed distribution. The estimation accuracy is additionally improved using an iterative procedure based on the dichotomous search of periodogram peak. Simulations carried out with several common noise distributions confirm the superiority of the proposed method over the standard one.**

***Keywords* — Dichotomous search, Fourier transform, frequency estimation, polynomial-phase signal.**

I. Introduction

The polynomial phase signals (PPSs) are observed in numerous research fields including radar, sonar, biomedicine, seismology [1]–[3]. Consequently, many methods for the phase coefficients estimation of the PPS have been proposed in the literature [3]–[8]. One of the most popular is the high-order ambiguity function (HAF), originally referred to as the polynomial-phase transform (PPT) [3], which represents the Fourier transform (FT) of the high-order instantaneous moment (HIM) of a PPS. Applying the HIM of the same order as the underlying PPS results in a complex sinusoid whose frequency is proportional to the highest-order phase coefficient, which, in turn, can be estimated using well developed sinusoid frequency estimation techniques [9]–[13]. Once the highest-order coefficient is estimated, the phase order of the PPS can be reduced by one by dechirping the PPS. This procedure is repeated until all coefficients are estimated.

The most popular sinusoid frequency estimation techniques are based on the search of the periodogram's maximum, an approach additionally facilitated by the fast FT (FFT) algorithms. The periodogram maximization is performed in two stages, coarse search and fine search. The coarse search represents finding the maximum bin of the FFT of a sinusoid corrupted by Gaussian noise. The fine search represents refining the coarse estimate through some iterative procedure [9]–[13].

When complex sinusoid is, however, corrupted by a heavy-tailed noise, the performance of the standard periodogram is severely degraded. Such a situation is encountered in the HIM, where the original noise, whether heavy-tailed or not, is transformed to a heavy-tailed one.

In this paper, we propose a method for the PPS phase estimation having in mind the heavy-tailed nature of the additive noise. To this end, we resort to methods based on the robust statistics [14]. In specific, we will use the robust *M*-periodogram [15], [16], where a non-quadratic loss function is used for fitting of observations corrupted by noise with unknown heavy-tailed distribution. In addition, we propose a fine search procedure that represents a modified version of the dichotomous search proposed in [11]. In contrast to the original dichotomous search, our modification does not require the zero-padding of the data, which is a very desirable property since the robust *M*-periodogram is calculated iteratively at each frequency.

Paper is organized as follows. Section II covers the theoretical background regarding the HAF. A brief overview of the robust *M*-periodogram is given in Section III, whereas the proposed fine search method is presented in Section IV. Simulations are given in Section V, and conclusions are drawn in Section VI.

II. High-order ambiguity function

The HIM of a signal $x(n)$ is defined as

$$\begin{aligned} &x_1(n) = x(n)\\ &x_2(n;\tau) = x_1(n+\tau)x_1^*(n-\tau)\\ &\dots\\ &x_P(n;\tau) = x_{P-1}(n+\tau)x_{P-1}^*(n-\tau), \end{aligned} \quad (1)$$

where τ is the time-lag. The HAF is defined as the FT of the HIM,

$$X_P(f;\tau) = \sum_{n=0}^{N-2P\tau-1} x_P(n;\tau)e^{-j2\pi fn} . \quad (2)$$

When $x(n)$ is a monocomponent Pth-order PPS, i.e.

$$x(n) = Ae^{j2\pi\sum_{m=0}^{P} a_m (n\Delta)^m} , \quad (3)$$

where A is constant real-valued amplitude, a_m are the phase coefficients and Δ is the sampling interval, the Pth-order HIM of $x(n)$ is a complex sinusoid with frequency [7]

$$f = 2^{P-1}\Delta^P P! a_P \tau^{P-1} . \quad (4)$$

The coefficient a_P can be therefore estimated by searching for the position of maximum in the HAF.

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If we dechirp $x(n)$ using the estimated value $\hat{a}_P$,

$$x'(n) = x(n)e^{-j2\pi \hat{a}_P (n\Delta)^P}, \qquad (5)$$

the resulting signal $x'(n)$ will be a (P-1)th-order PPS. Now the coefficient a_{P-1} can be estimated by searching for the position of maximum in the (P-1)th-order HIM of $x'(n)$. This procedure can be repeated to estimate all lower-order phase coefficients [3].

Consider, for example, the second-order PPS embedded in additive Gaussian noise $\nu(n)$,

$$y(n) = Ae^{j2\pi(a_1 n\Delta + a_2 n^2\Delta^2)} + \nu(n). \qquad (6)$$

The second-order HIM of $y(n)$ is

$$\begin{aligned} y_2(n;\tau) &= [Ae^{j2\pi(a_1(n+\tau)\Delta + a_2(n+\tau)^2\Delta^2)} + \nu(n+\tau)] \\ &\times [Ae^{-j2\pi(a_1(n-\tau)\Delta + a_2(n-\tau)^2\Delta^2)} + \nu^*(n-\tau)] \\ &= A^2 e^{j4\pi a_1 \tau\Delta} e^{j8\pi a_2 \tau\Delta^2 n} + \nu(n+\tau)\nu^*(n-\tau) \quad (7) \\ &+ Ae^{-j2\pi(a_1(n-\tau)\Delta + a_2(n-\tau)^2\Delta^2)}\nu(n+\tau) \\ &+ Ae^{j2\pi(a_1(n+\tau)\Delta + a_2(n+\tau)^2\Delta^2)}\nu^*(n-\tau). \end{aligned}$$

The first component of $y_2(n;\tau)$ is a complex sinusoid, whereas the other three are noise components. The last two noise components are Gaussian processes, while the first one, $\nu(n+\tau)\nu^*(n-\tau)$, has a heavy tailed distribution. Clearly, even in the second-order HIM, the frequency estimators based on the standard periodogram will not yield satisfactory results. The situation gets even worse with higher-order HIMs. A solution to this problem is presented in the sequel.

III. ROBUST M-PERIODOGRAM

Consider a complex sinusoid corrupted by heavy-tailed noise $\nu(n)$,

$$y(n) = Ae^{j(2\pi f_0 n\Delta + \varphi)} + \nu(n), \quad n = 0,1,\cdots,N-1, \qquad (8)$$

where A, f_0 and φ are unknown real-valued amplitude, frequency and phase, respectively, and N is the number of samples. We wish to estimate f_0.

The M-estimates f_0^M and C^M of the frequency and amplitude, respectively, are introduced as a solution to the following optimization problem [16]:

$$(f^M, C^M) = \arg\min_{C, f\in Q_f} J(f, C) \qquad (9)$$

where

$$\begin{aligned} &J(f,C) = \textstyle\sum_n \rho(n)\left[F(e_R(n)) + F(e_I(n))\right] \\ &e(n) = y(n) - Ce^{j2\pi f n\Delta} \\ &e_R(n) = \mathrm{Re}\{e(n)\}, \quad e_I(n) = \mathrm{Im}\{e(n)\} \\ &Q_f = \{f \mid -1/(2\Delta) < f < 1/(2\Delta), f \neq 0\}. \end{aligned} \qquad (10)$$

In (10), $\rho(n)$ is a non-negative window function and $F(x)$ is a convex non-negative loss function. Particular case $F(x)=x^2$ in (9) yields the standard periodogram [16].

Of particular importance is the absolute value loss function $F(x)=|x|$, which is optimal for two important classes on noise distributions [16],

- class of nonsingular distributions, i.e., when nothing is known about the noise distribution except that its p.d.f. $g(x)$ satisfies $g(x)>0$, and
- class of approximate exponential distributions,

$$g(x) = (1-\gamma)f_0(x) + \gamma f_1(x), \quad 0 < \gamma < 1, \qquad (11)$$

where $f_0(x)$ is the Laplace distribution and $f_1(x)$ is an arbitrary distribution.

The robust M-periodogram is defined as [16]

$$I_R(f) = J(0,0) - J(f, C(f)), \qquad (12)$$

where

$$\begin{aligned} &J(0,0) = \textstyle\sum_n \rho(n)\left[F(y_R(n)) + F(y_I(n))\right] \\ &y_R(n) = \mathrm{Re}\{y(n)\}, \quad y_I(n) = \mathrm{Im}\{y(n)\}, \end{aligned} \qquad (13)$$

and $C(f)$ is a minimizer of $J(f,C)$ provided a fixed value of f,

$$C(f) = \arg\min_C J(f,C). \qquad (14)$$

Given $C(f)$, the frequency can be estimated as

$$f^M = \arg\max_{f\in Q_f} I_R(f), \qquad (15)$$

and, in turn, the amplitude can be estimated as:

$$C^M = C(f^M). \qquad (16)$$

In [16], a recursive algorithm for solving (9) is also provided, which can be summarized as follows:

Step 0. Initialization

$$C^{(0)}(f) = \frac{1}{\sum_n \rho(n)} \sum_n \rho(n) z(n) e^{-j2\pi f n\Delta}$$

$$\gamma^{(0)}(n) = \rho(n) \left.\frac{F(e_R(n)) + F(e_I(n))}{e_R^2(n) + e_I^2(n)}\right|_{C=C^{(0)}(f)}.$$

Step 1. Set k=1 and iterate

$$C^{(k)}(f) = \frac{1}{\sum_n \gamma^{(k-1)}(n)} \sum_n \gamma^{(k-1)}(n) z(n) e^{-j2\pi f n\Delta}$$

$$\gamma^{(k)}(n) = \rho(n) \left.\frac{F(e_R(n)) + F(e_I(n))}{e_R^2(n) + e_I^2(n)}\right|_{C=C^{(k)}(f)}$$

$k = k+1$

until either of conditions

$$\frac{\left|C^{(k)}(f) - C^{(k-1)}(f)\right|}{\left|C^{(k-1)}(f)\right|} \le \eta$$

$$k \le K$$

is met, where η=0 and K are given.

Step 2. Setting the amplitude and periodogram

$$C(f) = C^{(k)}(f), \quad I_R(f) = J(0,0) - J(f, C(f)).$$

Experiments showed a good convergence of the algorithm for $F(x)=|x|$ [16]. Number of iterations, for $\eta = 0.1$, never exceeded 15 and it was usually 3-5.

The algorithm is repeated for each f from Q_f and the frequency is estimated as the position of the robust M-periodogram peak according to (15). In practice, the set Q_f is given by a grid of Fourier frequencies

$$Q_f = \left\{ f \mid f_k = \frac{1}{2N\Delta}k, k = -N+1, \cdots N-1 \right\}.$$

In this paper, the HIM of the corrupted PPS signal represents the input signal to the robust M-periodogram. Estimation of the PPS phase coefficients is then obtained by the periodogram maximization, performed through the coarse and fine search. The coarse search represents the finding the maximum bin of the robust M-periodogram and fine search will be explained in the following section.

IV. DICHOTOMOUS SEARCH

The dichotomous search of periodogram peak is proposed in [11]. It is a binary search method where the discrete FT (DFT) peak is located and the frequency estimation is adjusted toward the larger of two DFT coefficients from either side of the located peak. New DFT coefficient is calculated halfway between the peak and the larger coefficient. The position of the calculated coefficient represents improved frequency estimation over the initial one. This procedure is iterated Q times. A drawback to this method is the zero-padding of the data, to a length of at least 1.5N, needed to approach the Cramer-Rao bound (CRB) [12].

In [12], a modification of the dichotomous search is proposed; it attains the CRB without the zero-padding. We will use a similar approach that also does not require the zero-padding. The proposed method can be described as follows:

Step 0. Coarse search

Calculate the robust periodogram $I_R(f)$ at the grid of Fourier frequencies and find the frequency of periodogram's maximum, f_m. Denote $I_0=I_R(f_m)$. Calculate $I_R(f)$ at points $f_m \pm \Delta f/2$, i.e.

$$I_{-1} = I_R\left(f_m - \Delta f/2\right) \qquad I_1 = I_R\left(f_m + \Delta f/2\right)$$

where Δf is the frequency resolution.

Step 1. Iterations

Iterate Q times

$\Delta f = \Delta f/2$

if $I_1 > I_{-1}$ then $I_{-1} = I_0$ and $f_m = f_m + \Delta f$

else $I_1 = I_0$ and $f_m = f_m - \Delta f$

calculate $I_R(f_m)$ and set $I_0 = I_R(f_m)$,

Step 2. Final frequency estimation

Obtain the final frequency estimation as

$$f^M = f_m.$$

In each iteration, the frequency resolution Δf is halved. In [13], the authors propose an iterative calculation of the frequency displacement δ; in each iteration, δ is updated by a value that depends on the ratio of sum of $X_{-0.5}$ and $X_{0.5}$ and their difference, where $X_{-0.5}$ and $X_{0.5}$ are DFT bins displaced by $-\delta$ and δ from the current frequency estimate, respectively. This approach is, however, optimal for the Gaussian noise environment. In the dichotomous search, the only assumption regarding the peak shape is that it is a monotonically increasing function in interval $[f_t-\Delta f/2, f_t]$ and monotonically decreasing in $[f_t, f_t+\Delta f/2]$, where f_t is the true signal frequency.

V. SIMULATIONS

In the experiment, we consider the input signal

$$y(n) = x(n) + \nu(n), \quad n = 0,1,\cdots,N-1, \tag{17}$$

where $x(n)$ is a third-order PPS,

$$x(n) = Ae^{j2\pi\left(a_1 n\Delta + a_2 (n\Delta)^2 + a_3 (n\Delta)^3\right)}, \tag{18}$$

with coefficients $a_1=N/6$, $a_2=-N/5$ and $a_3=N/3$, $\Delta=1/N$, and $\nu(n)$ is complex additive noise,

$$\nu(n) = \nu_R(n) + j\nu_I(n),$$

where real and imaginary parts $\nu_R(n)$ and $\nu_I(n)$ are i.i.d. variables. We considered four noise models as follows:

- Gaussian noise with the variance b^2;
- $\nu(n)=b\xi_1^3(n)+jb\xi_2^3(n)$, where $\xi_1(n)$ and $\xi_2(n)$ are Gaussian $N(0,1)$ variables. Variables $b\xi_1^3(n)$ and $b\xi_2^3(n)$ have the probability density [16]

$$g(x) = \frac{1}{3b\sqrt{2\pi}} \left|x/b\right|^{-2/3} e^{-\left|x/b\right|^{2/3}/2}.$$

We will refer to this noise as the Gauss³ noise.

- Cauchy noise with the distribution

$$g(x) = \frac{1}{\pi}\frac{\gamma}{x^2+\gamma^2}.$$

- α-stable noise with characteristic function

$$\varphi(t,\alpha,\beta,\gamma,\delta) = e^{jt\delta - |\gamma t|^\alpha \left(1 - j\beta \operatorname{sgn}(t)\Phi\right)},$$

where α is the characteristic exponent, β the skewness, γ the scale parameter, δ the location parameter, sgn(·) the sign function and Φ is [17]

$$\Phi = \begin{cases} \tan\left(\pi\alpha/2\right) & \alpha \neq 1 \\ -2\log|t|/\pi & \alpha = 1. \end{cases}$$

We will herein consider symmetric α-stable noise ($\beta=0$) with $\alpha=0.5$ and $\delta=0$.

In the first example, we will compare the standard periodogram to the robust M-periodogram in the a_3 phase-parameter estimation. The comparison will be performed in terms of appearance of false spectral maximum, a situation that a spurious spectral peak exceeds the desired one, in the spectrum of the third-order HIM of $y(n)$.

Fig. 1(a) and (b) depict one realization of the normalized spectrum of the third-order HIM of $y(n)$ when standard and robust M-periodogram are used, respectively. Additive noise is the Gauss³ noise with $b=0.2$.

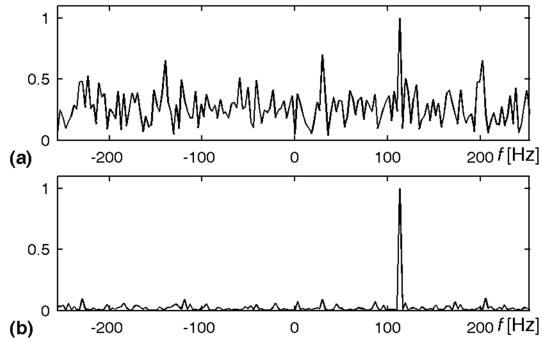

Fig. 1. (a) Standard periodogram and (b) robust M-periodogram of the third-order HIM of $y(n)$ (see (17)).

Fig. 2 presents false spectral maximum percentage (FSMP) versus noise parameter for all the considered noise types, indicating the superiority of the M-periodogram in terms of the peak resolvability. It is important to note that $F(x)=|x|$ is used for all the noise types except for the Gaussian noise, when $F(x)=|x|^{1.5}$ produced the best results. The FSMP is calculated over 1000 realizations of the considered methods.

In the second example, we will evaluate the performance of the fine search algorithm presented in

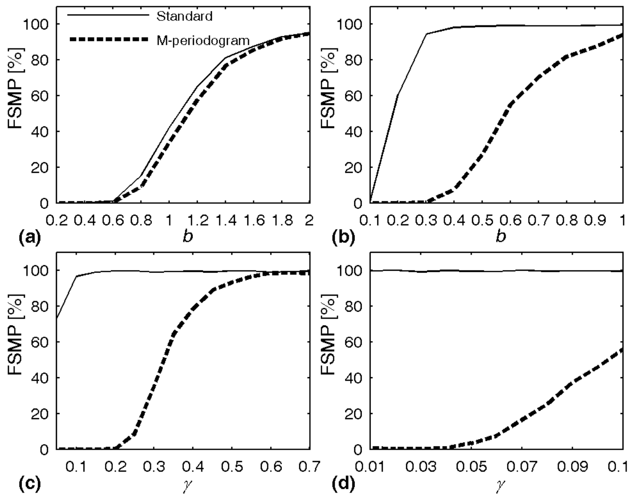

Fig. 2. False spectral maximum percentage versus noise parameter. (a) Gaussian noise; (b) Gauss^3 noise; (c) Cauchy noise; (d) α-stable noise.

section IV. To this end, we will compare the dichotomous search results to the oversampled robust *M*-periodogram in the estimation of all coefficients of $x(n)$ corrupted by the Gauss³ noise. Fig. 3(a), (b) and (c) present the mean squared error (MSE), versus the noise parameter, in the estimation of coefficients a_3, a_2 and a_1, respectively, where the MSE is calculated according to

$$\mathrm{MSE} = 10\log_{10}\frac{\sum_{k=1}^{N_{sim}}(a_j-\hat{a}_j)^2}{N_{sim}},\quad j=1,2,3,$$

where N_{sim} is the number of simulations. In this example, the oversampling factor N_{ov} is 10 and N_{sim}=1000.

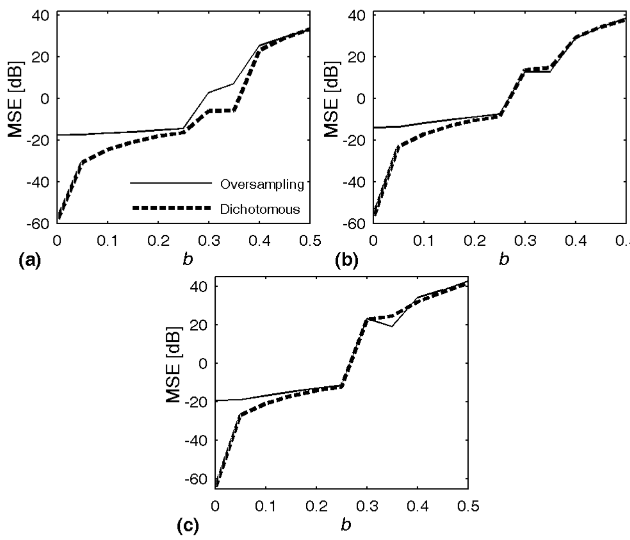

Fig. 3. Mean squared error versus noise parameter. (a) a_3 estimation; (b) a_2 estimation; (c) a_1 estimation. Gauss^3 noise is considered.

Clearly, the dichotomous search provides a significant improvement over the oversampling approach in terms of accuracy. In addition, its calculation complexity is much less, since, in each iteration, only one sample of the robust *M*-periodogram is calculated. On the other hand, the oversampling of the robust *M*-periodogram N_{ov} times implies N_{ov} times higher complexity, since it is calculated at each frequency iteratively. The power of FFT algorithms cannot be fully harnessed here.

VI. CONCLUSION

In this paper, we proposed a method for the estimation of PPS phase coefficients from the spectrum of the HIM. The method is based on the robust *M*-periodogram and it outperforms the standard periodogram-based approach in terms of appearance of spurious spectral peaks. In addition, in order to improve the estimation accuracy, we proposed the iterative procedure based on the dichotomous search of the robust *M*-periodogram. This approach outperforms the fine search based on the spectrum oversampling in terms of both the estimation accuracy and calculation complexity.

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